Inviscid Hypersonic Flow around a Semicone Body

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The steady hypersonic flow around a sharp-nosed slender semicone body is solved by applying the equivalence principle, which states that the flow is equivalent to a two-dimensional unsteady flow. It is assumed that the flow in the shock layer has constant density and homentropy. Two flowfields having different boundary shapes can be superposed to obtain a flowfield which satisfies the boundary condition on the body. The shock shape can be obtained from the mass conservation law equation in which the shock Mach number equals infinity. The result shows that the shock layer around the body is nearly uniformly thick, but that the shock wave is attached to the center of the planeside, i.e., the axis of the semicone body. The distribution of pressure on the body surface can be obtained from Bernoulli's equation for unsteady flow. It is also considered that the temperature in the shock layer can be obtained from the energy equation, and that the shock wave, in general, has acceleration in instances of two-dimensional unsteady flows.

Nomenclature

p = pressure
ρ = density
γ = ratio of specific heats
a = radius of the semicircular piston
τ = semiapex angle of the semicone body
q_r = velocity component in r-direction of the shock wave
U_∞ = freestream velocity
u_p = expansion speed of the semicircular piston
C_p = pressure coefficient
ϕ = velocity potential
ψ = stream function
W = complex potential
ζ = complex plane
U = strength of source and sink
v = fluid velocity
v_r = component of fluid velocity in r-direction
v_θ = component of fluid velocity in θ-direction
r^* = nondimensional radius
t = time
r = radius
S = area
q = velocity vector of the shock wave
T = temperature
e = internal energy per unit mass

Subscripts

∞ = freestream
s = behind the shock wave
p = on the semicircular piston or on the semicone body

I. Introduction

MANY studies on axisymmetric flows around sharp-nosed, slender bodies have been carried out. For the cone, especially, the exact equation and its numerical solution and many other approximate solutions have been obtained. However, there have not been many studies on asymmetric flows. Only flows near axisymmetry, for example, flows around a cone at a small angle of attack[1] or a half-cone body with a delta wing on it,[2] have been obtained analytically by perturbation theory. No investigations for complete asymmetric flows such as the flow around a semicone body have been reported.

In this paper a steady hypersonic flow around a semicone body, whose axis coincides with the direction of the freestream, is studied analytically by using the equivalence principle.[3] This principle, which converts steady hypersonic flows around slender bodies into corresponding two-dimensional unsteady flows, has been widely used. The corresponding two-dimensional flow is obtained by taking the time axis in the direction of the freestream. The flow around the semicone body is considered as a conical flow in which all the flow parameters are constant along rays from the vertex. Therefore, when it is changed into a two-dimensional unsteady flow, the flow conditions at any two times are similar. Consequently the whole flowfield around the semicone body can be obtained by determining the flow at a certain time driven by an expanding semicircular piston. In this analysis we assume that the fluid is a perfect gas which has neither viscosity nor heat conductivity and that the density and entropy are constant in the shock layer.

II. Analysis

The steady hypersonic flow around a sharp-nosed, slender semicone body is analyzed by changing it into a two-dimensional unsteady flow around a semicircular piston with the equivalence principle. The terms "semicone body" and "semicircular piston" refer to the three-dimensional steady flow and the two-dimensional unsteady flow, respectively.

Since the two-dimensional flow around the semicircular piston is symmetric with respect to the axis OX, we can examine the boundary shape as shown in Fig. 1. It is most convenient to use the polar coordinates (r,θ) in analyzing the flow. Point A is the origin, and A′ and C′ are at infinity. The arc $\widehat{BC}$ increases its radius at a constant speed u_p. The radius at any moment is $u_p t$. In this problem the fluid has constant density, and the flow is homentropic—that is irrotational' therefore, the equation which express the flowfield in the shock layer is Laplace's equation.

A. Velocity Field

The flowfield in the shock layer must satisfy the boundary conditions both on the piston and behind the shock wave. First, a flowfield which satisfies the boundary condition on the piston and is at rest at infinity shall be considered. Here, the existence of the shock wave near the piston is previously assumed. This flow can be uniquely determined. From this flowfield, the shock shape, which satisfies the boundary condition behind the shock wave, can be obtained. For changing

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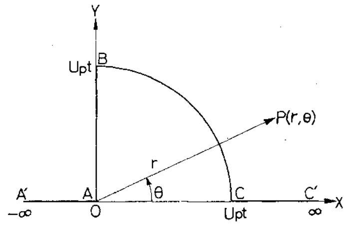

Fig. 1 Boundary shape and coordinate system.

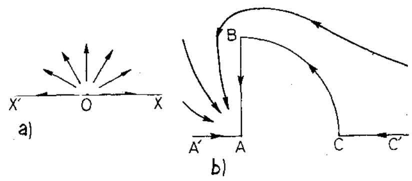

Fig. 2 ζ-plane a) flow due to a source, and b) flow due to a sink.

the flowfield into the three-dimensional steady hypersonic flow around the semicone body, the condition that the semiapex angle of the body is small must be satisfied.

The boundary conditions which must be satisfied by the previous flowfield are

on BC	$\partial\phi/\partial n = u_p$	(1a)
on AB, AA′ and CC′	$\partial\phi/\partial n = 0$	(1b)
at infinity	$\partial\phi/\partial n = 0$	(1c)
and point A must be a stagnation point		(1d)

where ϕ is the velocity potential and $\partial/\partial n$ is a normal derivative. The problem is reduced to solving Laplace's equation having similar boundary conditions, however, in two-dimensional problems there are some cases in which the solutions can be obtained without solving the equation. In this case, the solution can be obtained as follows.

Consider two flows on the complex plane ζ, as shown in Fig. 2 a) and b). The flow in a) has a source strength of 2U at the origin. The flow in b) has a sink strength of $-$U at the origin A, and the same boundary shape as in Fig. 1, where X, X′ are at infinity. By superposing b) and a) and fixing A to O, C′ to X, and A′ to X′, the flowfield which satisfies the boundary conditions in Eqs. (1) can be obtained. If W_1 and W_2 are the complex potentials in a) and b), respectively, the required complex potential W is

$$W = W_1 + W_2 \tag{2}$$

W_2 can be obtained by transforming the flowfield in the ζ-plane in Fig. 2, b) to one due to a sink in the upper plane of the complex plane.[4] From the assumption that the piston expands at a constant speed u_p, the strength of the source U is $\frac{1}{2}u_p^2 t$. Consequently, W becomes

$$W = \tfrac{1}{2}u_p^2 t\left\{2\log\zeta - \log\left[\frac{2-\mathfrak{A}}{-1-\mathfrak{A}}\right]\right\} \tag{3}$$

where

$$\mathfrak{A} = \left[\frac{1+i\zeta/a}{1-i\zeta/a}\right]^{2/3} - \left[\frac{1-i\zeta/a}{1+i\zeta/a}\right]^{2/3}$$

where the first term on the right-hand side is W and the second term is W_2.

Here, the nondimensional radius r^* can be introduced:

$$r^* = r/a \tag{4}$$

where r is distance from the origin and a is the radius of the piston. With this variable r^* the velocity potential and the stream function are expressed as

$$\phi = \tfrac{1}{2}u_p^2 t\left\{2\log r^* + 2\log a - \log\frac{[\{(F-2)(F+1)+G^2\}^2+9G^2]^{1/2}}{(F+1)^2+G^2}\right\} \tag{5}$$

$$\psi = \tfrac{1}{2}u_p^2 t\left\{2\theta - \left(\tan^{-1}\frac{3G}{(F-2)(F+1)+G^2}\right)\right\} \tag{6}$$

where

$$F = (R^2 + (1/R^2))\cos 2\alpha, \quad G = (R^2 - (1/R^2))\sin 2\alpha$$

$$R = \left[\frac{1+r^{*2}-2r^*\sin\theta}{1+r^{*2}+2r^*\sin\theta}\right]^{1/6}$$

$$\alpha = \tfrac{1}{3}\tan^{-1}\frac{2r^*\cos\theta}{1-r^*}$$

The velocity components in the r- and θ-directions are

$$v_r = \frac{u_p}{2r^*}\left\{2-\left(R+\frac{1}{R}\right)\cos\alpha\right\} \tag{7a}$$

$$v_\theta = \frac{u_p}{2r^*}\left(R-\frac{1}{R}\right)\sin\alpha \tag{7b}$$

Because v_r and v_θ are expressed only in terms of constants and the nondimensional variable r^* and θ, the velocities are always the same at two corresponding points. Although ψ increases in proportion to t, the shapes of the streamlines are always similar. This fact has already been stated in the

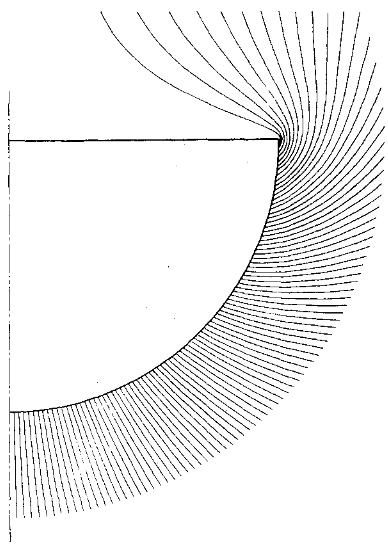
Fig. 3 Streamlines around a semicircular piston.

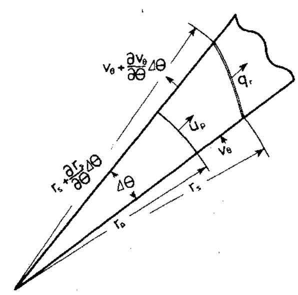
Fig. 4 Control surface.

equation of velocities. However, ϕ includes a function of time, (log a), which cannot be made nondimensional.

The streamlines at a certain moment are shown in Fig. 3. Since the flow is unsteady, the streamlines, which are envelope curves of the velocity vectors do not coincide with the paths of the fluid particles.

B. Determination of the Shock Shape

In the limiting hypersonic process, the pressure p_∞ and sound velocity a_∞ of the freestream become zero, and the shock Mach number becomes infinity. In this case, the density ratio across the shock wave is[5]

$$\rho_s/\rho_\infty = (\gamma+1)/(\gamma-1) \tag{8}$$

where γ is the ratio of the specific heats and is 1.4 for air.

To obtain an equation for the shock shape, apply the mass conservation law on the control surface shown in Fig. 4:

$$\frac{\partial}{\partial t}(\rho_s S) + \left\{1-\frac{\gamma-1}{\gamma+1}\right\}\rho_s q_r r_s \Delta\theta = \rho_s u_p r_p \Delta\theta + \rho_s \int_{r_p}^{r_s} v_\theta dr - \rho_s \int_{r_p}^{r_s+(\partial r_s/\partial\theta)\Delta\theta} \left(v_\theta + \frac{\partial v_\theta}{\partial \theta}\Delta\theta\right) dr \tag{9}$$

where S is the area of the control surface. The first term on the left-hand side is the rate of increase of mass on the control surface; the second term is the imaginary rate of increase of mass by the movement of the shock wave. The first term on the right-hand side is the mass per unit time swept by the moving piston with the velocity u_p, which is due to W_1: the second and third terms are the mass transfer onto the control surface by the component of flow velocity in the direction of θ, which are due to W_2.

For simplicity, consider the case where $t=1$. Since the distance of the shock wave from the origin r_s is equal to the shock velocity component in the r-direction q_r and $\partial/\partial t)\rho_s S)=0$, if terms higher than second order in $\Delta\theta$ are neglected and the formula is changed a little, it becomes the ordinary differential equation

$$v_{\theta s}\frac{dr_s}{d\theta} + \left\{1-\frac{\gamma-1}{\gamma+1}\right\}r_s^2 - v_{rs}r_s = 0 \tag{10}$$

where the velocity components v_{rs} and $v_{\theta s}$ just behind the shock wave have already been obtained in Eq. (7). Although the body shape affects these components, it does not appear explicitly in this equation, which can, therefore, be used for the shock shape around any body. The assumption that the flowfield has constant density and entropy is essential to the equation. The integration of Eq. (10) involves a constant. The shock wave is considered to intersect the axis of symmetry OX at a right angle at the point $\theta=0$. Since $v_{\theta s}=0$ at this point, the continuity equation can be considered in the radial direction only

$$\left\{1-\frac{\gamma-1}{\gamma+1}\right\}r_s = v_{rs} \tag{11}$$

Let us integrate Eq. (10) by making Eq. (11) the initial value of the integration. The shape of the shock wave for $\gamma=1.4$ is shown in Fig. 5, where r_s is $1.081126\times r_p$ at $\theta=0$. The shock wave attaches to the piston at its axis of symmetry because this is a stagnation point for the circumferential flow, but no freestream fluid is entrained.

The paths of the fluid particles, illustrated in dimensionless form, are also shown in Fig. 5. This illustrates the relative positions of fluid particles between the piston and the shock wave, but not the actual particle paths. It is shown that the fluid particles pass through the shock wave and approach the body surface gradually.

The streamlines and the particle paths shown in Fig. 3 and 5, respectively, are for a two-dimensional unsteady flow around a semicircular piston. Changing this flowfield into the three-dimensional steady flow around the semicone body, the streamlines, which coincide with the particle paths in this case, are shown in Fig. 6. These results are obtained by adding the freestream velocity U_∞ in the direction of the axis of the semicone body to the velocity vectors of the two-dimensional flow and integrating, because the velocity component in the direction of the axis is approximately equal to U_∞ as long as u_p/U_∞ is small. Here U_∞ can be taken independently of u_p. The cross-section AA shows the place where the fluid particles enter the shock layer. Figure 5 can also be considered as the projection on a certain cross section of the streamlines around the semicone body. The points a,b,c,d,e of Figs. 5 and 6 correspond.

C. Pressure Coefficient

Since the flow around the semicircular piston is unsteady, Bernoulli's equation for unsteady flow is used to obtain the pressure distribution. The equation is[6]

$$\partial\phi/\partial t + p/\rho + \tfrac{1}{2}v^2 = C \tag{12}$$

Here, from Eq. (5)

$$\partial\phi/\partial t = \tfrac{1}{2}u_p^2 t(\phi^* - r^*\partial\phi^*/\partial r^* + 2 + 2\log a) \tag{13}$$

where ϕ^*, a function of the nondimensional variable r^* and θ, is

$$\phi^* = 2\log r^* - \log\left[\frac{\{(F-2)(F+1)+G^2\}^2+9G^2}{(F+1)^2+G^2}\right]^{1/2} \tag{14}$$

The constant in Eq. (12) can be obtained from the condition at the shock wave. In this problem, since it is assumed that the

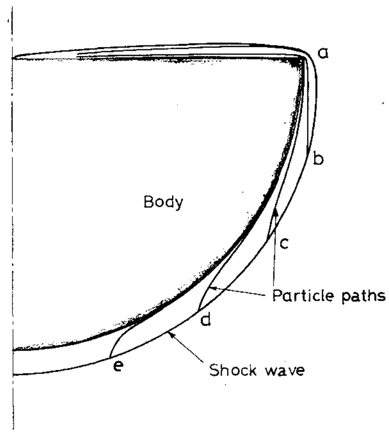

Fig. 5 Shock shape and particle paths.

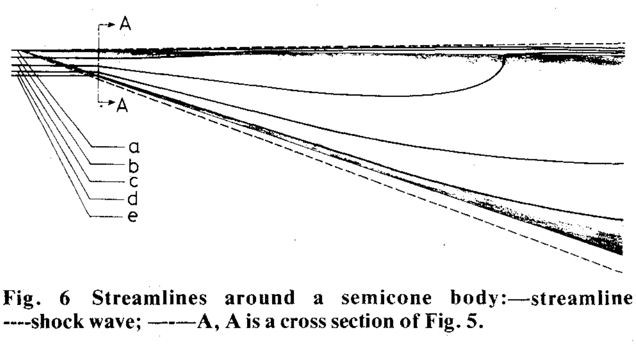

Fig. 6 Streamlines around a semicone body:—streamline; ----shock wave; ——A, A is a cross section of Fig. 5.

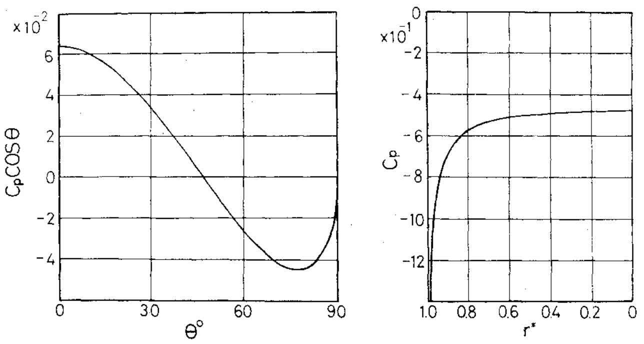

Fig. 7 Pressure distribution on the surface of a cross section.

flow is homentropic in the shock layer, C is constant not only along the streamline but also everywhere in that layer. C can be determined from the condition at the point on the axis of symmetry; i.e., $\theta=0$. From the law of conservation of momentum we obtain

$$p_s=\rho_s v_{rs}(r_s-v_{rs}) \tag{15}$$

Therefore, the constant C becomes

$$C=\tfrac{1}{2}u_p^2(C^*+2+2\log a) \tag{16}$$

where C^* is expressed as

$$C^*=\left[\phi^*-r^*\frac{\partial\phi^*}{\partial r^*}\right]_s+2\left[\frac{v_{rs}}{u_p}\right]\left[\frac{q_r}{u_p}\right]-\left[\frac{v_{rs}}{u_p}\right]^2$$

Consequently the pressure becomes

$$p=\tfrac{1}{2}u_p^2\rho_s\left\{C^*-\phi^*+r^*\frac{\partial\phi^*}{\partial r^*}-\left[\frac{v}{u_p}\right]^2\right\} \tag{17}$$

Considering the flow around the semicone body, the pressure coefficient becomes

$$C_p=\left[\frac{\rho_s}{\rho_\infty}\right]\left[\frac{u_p}{U_\infty}\right]^2\left\{C^*-\phi^*+r^*\frac{\partial\phi^*}{\partial r^*}-\left[\frac{v}{u_p}\right]^2\right\} \tag{18}$$

u_p/U_∞ corresponds to the semiapex angle of the semicone body. Therefore Eq. (18) can be written as

$$C_p=\frac{\gamma+1}{\gamma-1}\tan^2\tau\left\{C^*-\phi^*+r^*\frac{\partial\phi^*}{\partial r^*}-\left[\frac{v}{u_p}\right]^2\right\} \tag{19}$$

The distribution of the pressure coefficient on the body surface at a cross section is shown in Fig. 7.

III. Discussion

The discussion here refers to the two-dimensional unsteady flow around the semicircular piston.

A. Constant Density

Although it is assumed that the density in the shock layer is constant, near the corner where the coneside and the planeside meet, the pressure becomes so low that the effect of density variations should not be overlooked. As the fluid flows from the coneside onto the planeside, it expands in volume. However, this assumption not only gives a good approximation itself on the coneside but also can be used as the first approximation for higher approximations.

B. Homentropy and Velocity of Shock Wave

The assumption that the entropy is uniform in the shock layer is not exactly correct because the strength of the shock wave around the semicircular piston varies with θ. However, as shown in Fig. 5, the shock wave on the coneside has an almost uniform strength and the fluid from it flows isentropically onto the planeside. On the other hand, fluid containing strong vorticity flows near the shock wave from the vicinity of the shoulder to the planeside. This indicates that the shock layer can be thought of as being homentropic with the exception of a thin vortical layer near the shock wave on the planeside.

In the case of an asymmetric flow, the shock wave generally has some acceleration. Only when the shock wave expands strictly in the radial direction is the shock velocity constant, assuming that the shock wave radius increases in proportion to time. The shock velocity q is $q=K_s/\partial F/\partial r$ when the shock shape is expressed as $F(r,\theta)$, and the center of similarity coincides with the origin, where K_s is a constant. If it is assumed that the shock wave expands in the normal direction, the velocity is expressed as $q=K_s\,\mathrm{grad}F/|\mathrm{grad}F|^2$ and the shock wave in this case has acceleration.

C. Temperature

From the assumptions that density is constant and that heat conductivity and radiation can be neglected, the temperature in the shock layer is also constant. For a perfect gas, the temperature can be obtained from the energy equation as follows. The energy equation is written as

$$\int_{S_s}\rho_s(e_s+\tfrac{1}{2}v^2)\,dS=\int_{S_s+S_p}(\rho_\infty e_\infty)\,dS+\bar{W} \tag{20}$$

It becomes

$$\rho_s c_v T_s S_s-\tfrac{1}{2}\rho_s\int_\ell\phi(\partial\phi/\partial n)\,d\ell=\rho_\infty c_v T_\infty(S_s+S_p)+\bar{W} \tag{21}$$

where S_s is the area of the shock layer, ℓ is the closed curve which surrounds the area, S_p is the area of the piston, c_v is the specific heat at constant volume, and $\bar{W}$ is the work done by the piston which can be obtained from the pressure distribution on the piston surface. In the case of limiting hypersonic flow where $T_\infty\mapsto 0$, T_s can be obtained easily.

D. Superposition of Solutions

The method of superposition of solutions which is used in this paper is not only very effective in solving a two-dimensional unsteady flow of constant density and homentropy but also the ground work for the higher approximation of a flow considering density change and viscosity.

IV. Conclusion

By superposition of solutions, the two-dimensional unsteady flow around a semicircular piston is solved. The shock shape is obtained from this flowfield when the shock Mach number is infinity. The shock radius is maximum at $\theta=0$ on the coneside and is attached to the piston at $\theta=\pi$.

The streamlines and shock shape of the flow around a semicone body are shown, which are obtained from the two-dimensional unsteady flow around a semicircular piston by using the equivalence principle. The shock angle at $\theta=0$, where it is largest, is a little smaller than that for a circular cone with the same semiapex angle at zero yaw. When the semiapex angle is 10° the shock angle of the semicone body is 10.79° in comparison with 11.30° for the circular cone.[7]

The pressure is obtained from Bernoulli's equation for unsteady flow. The pressure coefficient at $\theta=0$ on the body surface is also smaller than that for a circular cone.

References

[1]Cheng, H. K., "Hypersonic Flows Past a Yawed Circular Cone and Other Pointed Bodies," *Journal of Fluid Mechanics,* Vol. 12, Pt. 1, 1962, pp. 169-191.

[2]Mandl, P., "Theoretical Study of the Inviscid Hypersonic Flow about a Conical Flat-Top Wing-Body Combination," *AIAA Journal*, Vol. 2, Nov. 1964, pp. 1956-1964.

[3]Chernyi, G. G., *Introduction to Hypersonic Flow*, Academic Press, New York, 1961, p. 72.

[4]Milne-Thomson, L. M., *Theoretical Hydrodynamics*, 5th ed., Macmillan, London, 1968, pp. 287-288.

[5]Liepmann, H. W. and Roshko, A., *Elements of Gasdynamics*, Wiley, New York, 1960, p. 65.

[6]Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967, p. 383.

[7]Sims, J. L., "Tables for Supersonic Flow around Right Circular Cones," NASA, SP-3004, 1964.